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Classical and Neoclassical harmonies and dissonances

Paul A. Samuelson

'For a man with a hammer, everything looks like a nail'.
Warren Buffett
'Where it is a duty to worship the sun, the laws of heat will be poorly understood'.
John Morley
'Where it is a duty to *abhor* the sun, the laws of heat will be poorly understood'.
Paul Samuelson

1. Prologue

Possessing an idiosyncratic antipathy to adversary procedures in scientific discourse, I intend here to present a low-key, candid sample of my takes on heterogeneous capital competitive models *for non-neoclassical limited-substitutability convex technologies.* Just as to understand one country one needs to know two (or more) countries, I will be repeatedly comparing and contrasting neoclassical paradigms with earlier century classical paradigms and with my understanding of post Leontief-Sraffa paradigms.

In advance I want to honour Joan Robinson (1956) and Piero Sraffa (1926, 1960) for their seminal questioning of mainstream economists' complacencies and normative dogma about intertemporal capital theory.[1]

Address for correspondence
Department of Economics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA; e-mail: jamu@mit.edu

1 It was I who had advised the Cambridge Press to definitely publish an American edition of the 1960 Sraffa classic. It tells us something about the vagaries of fashion in a science's evolution that demand for Sraffa (1960) has by now so dried up as to force it out of print.

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Later I will list some personal indebtedness to modern non-mainstream economists.

The following text is, by agreement, not peer reviewed. So let every reader be on notice that errors and infelicities may be present. To optimize the rationed space allotted to me, I skip formal proofs and make no attempt to integrate optimally discussions of the several different topics addressed.

My ordering of different topics is neither related to their relative importance nor to their chronological provenance.

2. Introduction

Here are a few points that the present analysis will try to explicate.

1. It is a myth that there ever did exist a plausible classical paradigm in which competitive price ratios—among deer, beaver, corn and rye—were invariant under changes in objective consumers' demand tastes.
2. Also it is textually dubious that post-1870 neoclassicism (Jevons, Menger, Walras, Wicksell, Wicksteed, Marshall, Edgeworth, Cassel, Ramsey, Hicks, Meade, Solow, Samuelson, Arrow, Debreu . . .) differed from classicism (such as in Cantillon, Hume, Turgot, Smith, Malthus, Ricardo, James and J.S. Mill . . .) importantly because the former linked purely competitive supply and demand with *constant returns to scale* whereas the earlier group definitely did not have to do so.

The limited invited space here will be used analytically, not textually. Readers can consult such excellent commentators as Schumpeter (1954), Blaug (1978), Hollander (1987) or Niehans (1990). My own teachers in and out of the classroom were Viner, Knight, Taussig, Cannan, Robbins and numerous others. My views on these matters, as explicated analytically here, do happen to mostly agree with *their* views. But our common views should carry no weight in present debates. Today's purpose is to *deduce* what are correct behaviour equations under well-specified scenarios. If anywhere my non-peer-reviewed syllogisms are found to be erroneous, the present exercise will have been valuable in helping establish where the truth does *probably* reside.

3. In my many, many dialogues with Professor Joan Robinson, we worried about the normative properties of supply – demand markets. What she deemed to be apologetics for too-fat capitalists, I took to be solvable

problems about 'intertemporal Pareto optimalities or *non*-optimalities'.[2]

a) Does 'double switching' imply intertemporal Pareto *non*-optimality? (see Pasinetti *et al.*, 1966). Does 'capital', reversibility?
b) Does the 1956 Ruth Cohen *curiosium* (see Robinson 1956: 109–10) and the Liviatan and Samuelson (1969) violation of the 'normal' Ricardo-Hollander *inverse trade-off between the real wage rate and the interest rate* imply a similar non-optimality? If so, the Samuelson and Etula (2006b) violation would be also non-Pareto optimal. Since my space is so limited, I will simply report here that the 1956, 1969 and 2006 (so-called) anomalies are provably intertemporally Pareto optimal.
c) What about a view that *only stationary* states are deductively tractable? My use here of twenty-first century *dynamic* Samuelson-Etula Master Functions will rebut that claim for heterogeneous-capital scenarios (as was done in Samuelson and Etula (2006a) and in the Samuelson and Etula (2006c) *divertimento*-sonnet for Graz's sixtieth birthday Fest for Heinz Kurz). Demonstrated here will be some generic problematics about stationary states.
d) Buffett's above quoted quip can apply to the definable twenty-first century dynamic Master Function $C_1(t+1) = M[K_1(t), K_2(t); K_1(t+1), K_2(t+1) + C_2(t+1)]$ 'hammer', which deduces for Leontief-Sraffa *limited substitutability technologies* much the same *qualitative* properties as will hold for J.B. Clark-Ramsey-Solow *neoclassical* technologies. In particular, *non*-spurious

2 Early on I would shift conversations away from present-day mixed economies. She had become impatient with the Senior-Böhm-Fisher view that, even in the absence of Schumpeterian innovations, generalized accumulation of capitals by motivated saving decisions to sacrifice some of today's consumption in trade-off for more permanent future consumption could raise real wage rates while lowering safe interest rates. She dismissed that as leak-down flap-doodle concocted by apologists for capitalism. Successively, she came to admire Leninism-Stalinism, Castroism, Maoism and in the end North Koreanism. Therefore, I would shift analytic discussion away from contemporary economic history. Innocently, I would ask: 'Joan, how should Mao act to elevate China's real per capita incomes?' Without hesitation, she would reply: 'First he must select the investment projects with the highest relative yields. That done, go on to further projects with lower yields'. Sweet it was to be able to agree on some things. Neither Joan nor Piero ever bothered to rebut in print the pre-1935 neoclassical versions of capital deepening of Ramsey (1928). When I challenged her in this regard by describing single K—contemptible 'Leets'—she lost interest. Nor did my citing of the heterogeneous Ramsey-type neoclassical scenario in Samuelson and Solow (1956) or Samuelson (1960) pique her interest.

> marginalisms can be definable for *both* of these technologies—so that heterogeneous factors can (almost everywhere) have respective (marginal productivity!) equilibrium yields equal to ∂ output/∂ input or $\Delta Q/\Delta K_j$ expressions. Given more space, my expositions could have been more complete and less intuitive.

Supply without demand is like one hand clapping. Microscopic examination of Sraffa's (1960) 100 pages will detect little discussion of how a shift of consumer tastes away from durable goods might influence equilibrium profit rates. Nor do those pages contain nuanced analyses of how Robinson Crusoe's time preference for corn today rather than next year might alter materially equilibrium interest rates.

Linear programming paradigms à la George Dantzig (1948) applied to intertemporal scenarios shout out the need for tastes-demand equations to provide the *complete* equations of dynamic and static competitive equilibrium. Tersely—too tersely—I touch upon this vital problem.

Samuelson (1966) expressed sincere gratitude to Sraffa, Robinson, Garegnani, Pasinetti, Bruno-Burmeister-Sheshinsky, Kurz-Salvadori, Bliss, Schefold, Metcalfe-Steedman, Morishima and many others who corrected my earlier errors prior to and post publications on the complexities of intertemporal economics. In my considered opinion, an early Nobel Prize shared by Robinson – Sraffa – Harrod would have added lustre to Stockholm's first-decade choices.

3. Why 'natural prices' cannot be defined in the 1750 – 2006 era

For a few pages only, Smith (1776) exposited the Labour Theory of Value simpliciter. Suppose that to produce $q_1 = 1$ beaver, three units of L_1 Labour were needed; but to produce $q_2 = 1$ of deer, ten units of L_2 Labour were needed. Then Smith could cogently deduce:

$$P_2/P_1 = (10 \text{ of } L_1)/(3 \text{ of } L_2) = 3\,\tfrac{1}{3}, \text{ independently of tastes.} \qquad (1)$$

This 'natural price' would hold true whatever might be the volatility of changing consumer tastes for the goods. Thus, when everyone always spends ninety percent of income on beaver consumption and ten percent on deer consumption, 3 1/3 would hold; 3 1/3 would also still hold if tastes changed so that all spent fifty – fifty percent on those goods. What holds for these two goods would hold also for three goods (deer, beaver, quail) or for N goods if all were producible out of Labour alone with constant unit labour costs *independently of scale*.

Suppose Smith also had reported that each $q_3 = 1$ of corn needs only two units of (homogeneous) Acres of Land, $A_3 = 2$; and that each $q_4 = 1$ of sugar needs exactly $A_4 =$ ten units of Land. Then canny Smith, soft-pedalling the Labour Theory of Value, might sign up for the Cantillon-Henry George, Samuelson (1959) 'Dated-Land-Content Theory of Value':

$$P_3/P_4 = 1 \text{ Acre}/10 \text{ Acres} = 0.1, \text{ independently of tastes.} \tag{2}$$

How would Smith, the embryonic general-equilibrium theorist, deduce competitive price ratios *for all these four goods at a time*? Not yet could he do better than

$$\begin{aligned}(P_2/P_1; P_3/P_1, P_4/P_1) &= \left(3\,\tfrac{1}{3}; ?; ?\right), \text{or}\\ (P_1/P_3, P_2/P_3; P_4/P_3) &= \left(?, ?; \tfrac{1}{2}\right).\end{aligned} \tag{3}$$

What Smith still lacks among other things is the (Land Rent)/(Worker Wage) ratio $= R/W$. Until economists Smith or Ricardo know *Distribution* they generically cannot know *Values*. And vice versa! It is a circle, but it can be a virtuous general equilibrium circle. Also, do note that both Equations (1) and (2) do indeed obey *constant returns to scale*.[3]

The equations missing are what the Sraffa I knew never seemed to like very much: demand tastes, volatile as they sometimes are. So let us skip back to young J.S. Mill when he was successfully completing and perfecting Ricardo's (1817) comparative advantage trade of Portugal's wine for England's cloth. Eschewing the metaphysical fuzziness of marginal utilities, John Bull Mill *objectively* could postulate that (say) we all spend twenty-five percent of our disposable incomes on each of $(q_1, q_2; q_3, q_4)$, denoted by

$$\hat{m}_j = p_j q_j / \sum_1^4 p_i q_i = \hat{\tfrac{1}{4}},\ j = 1, 2, 3, 4 \tag{4}$$

We still, however, will be missing needed essential data on available exogenous supplies of total Labour and total Acres of Land: here are such $\hat{L}$ and $\hat{A}$ data:

$$\begin{aligned}\text{exogenous } \hat{L} &= L_1 + L_2 = \text{ say } \hat{100}\\ \text{exogenous } \hat{A} &= A_3 + A_4 = \text{ say } \hat{50}\end{aligned} \tag{5}$$

All my exogenous numerical data have been put into bold type.

3 Suppose Sraffa lets Smith posit *increasing* scales returns for beaver: $q_1 = L_1^2$ say. And let him posit *decreasing* scales returns for deer: $q_2 = \sqrt{L_2}$ say. From that quagmire no 1776 or 1926 or 2007 economist can infer a coherent or plausible competitive P_i/P_j formula.

Given Equations (4) and (5), *all* the unique classical competitive equilibrium real prices (factor and goods) are determinate and calculable by simplistic linear substitutions. Smith, Ricardo and any reader should be up to the task. Do do it. And see how by $2+2=4$ arithmetic, *all* the (non-natural) real prices are altered by changes in demand tastes and in relative factor supplies.

To save space, I will make the same point about the *generic* impossibility of natural classical prices by replacing the above 4-good scenario with a terser 2-good scenario involving only goods 1 and 3. All exogenous parameters remain the same, except that now Mill has consumers spending fifty:fifty percent on labour-produced q_1 and on land-produced q_3. We will still be left with two ratios, two intrinsically non-natural real prices. Here in a nutshell is their demonstrated fatal lack in invariance.

Write $\hat{m}_3 = \hat{m}_1 = \frac{1}{2}$ for Mill's exogenous p_3q_3/p_1q_1 ratio of expenditures. Write $\hat{L}/\hat{A}$ for exogenous relative factor supplies. And write $\hat{\Pi}_1 = \hat{3}$ and $\hat{\Pi}_3 = \hat{1}$ for respective technical cost coefficients. Then:

$$\begin{aligned} R/W^* &= (\hat{m}_3/\hat{m}_1)\ [\hat{L}/\hat{A}]; (P_3/P_1)^* = [\hat{\Pi}_1/\hat{\Pi}_3](R/W)^* \\ &= [\hat{\Pi}_1/\hat{\Pi}_3](\hat{m}_3/\hat{m}_1)[\hat{L}/\hat{A}] \end{aligned} \tag{6}$$

Equation (6) is the QED for how 'unnatural' classical prices had to be *generically*. Only singularly—implausibly singularly—could changes in $\hat{m}_3/\hat{m}_1$ tastes or in $\hat{L}/\hat{A}$ factor supplies leave intact the natural prices nominated in Sraffa (1926) and 'approximated' by Stigler (1958). The point is so simple as to be almost banal, were it not for its prolonged neglect in the commentator literature.

4. Those occasional classical cases where goods might be *a*temporally producible by fixed proportion 'doses' of labours and land

Sometimes when early scholars did not know how to impute the separate shares of Labour and Land, they would posit that 1Q of Corn might require, say, a 'dose' of 2 from Labour *cum* three acres from Land. It was not hard to realize that in such a *single* activity case, no determinate fractional sharing of the harvest between landowner and labourers could come solely from the side of technology and costs. Extraneous demand or supply relations might cut the Gordian Knot at any fractional point between zero percent to labour wages and 100 percent to land rents or 100 percent to Labour and zero percent to Land.

Still, both in the early Anglo-Saxon and Germanic literature, there grew up a fairly sophisticated understanding of joint products or joint inputs. This is why it will be instructive for my future *temporal*-economic discussions of *heterogenous* capitals, first to work out here the easier to understand *a*temporal marginalist scenarios where a Q of corn output gets produced by doses of, say, (homogeneous Land; homogeneous male Labour, homogeneous female Labour), denoted by $(X_0;X_1,X_2)$. 'Threeness' is important pedagogically. Why? Because Sraffa's (1960) temporal analysis of Labour and Wheat and Iron does have the three-ness that Labour *cum* scalar K would lack. I will explicate how a Master Function is definable (in general!) for $Q(t) = F[X_0(t);X_1(t),X_2(t)] \equiv$ for short $F[X_0;X_1,X_2] = Q$. It will then remain only a short step to explain how more complicated Master Functions can apply both to everywhere differentiable Clarkian production functions and also (!) to limited substitutability, non-neoclassical, Leontief-Sraffa production functions for intertemporal input/output relations.

Given three exogenous atemporal Xs, determinate sharing of Q between them will be attained only when all three Xs *attain* supply and demand equilibrium distributive prices, (land rent, real male wage, real female wage) $\equiv \left(y_0^*, y_1^*, y_2^*\right)$, such that the three distributive shares will then be:

$$\frac{y_0X_0}{Q} + \frac{y_1X_1}{Q} + \frac{y_2X_2}{Q} = 1 \qquad (7)$$

Only with scale-returns *constancy* will this addition to unity obtain. It will still remain an impossible puzzle when only a single A dose is technologically known. But distribution may become definitely unpuzzled when *three* sufficiently different A or B or C usable doses are known to every would-be competitive entrant into the Q industry.

Table 1 summarizes succinctly the following known-to-all A, B, C and D alternative sub-techniques.

Instead of employing the usual Leontief-Sraffa input/output coefficients of the form $a_{\text{land,corn}} =$ land input/corn output, Table 1 presents the equivalent technical data *normalized* to *unity* acres of the (homogeneous) Land. Readers should be grateful for this numeraire convention, because it minimizes diagrammatic excursions into the third dimension. Instead, the $[X_1/X_0,X_2/X_0]$ Euclidean plane can, by means of definable triangles or polygons, convey to the eye the whole intuitive story. Also, I put plentiful 0s and 1s in Table 1 as a crutch to inexperienced readers. Feel free to add $= \pm\ 0.001$ to 1's singular 0 coefficients, thereby altering quantitative results by itsy-bitsies only. (For Clark-Douglas neoclassical disciples, their everywhere differentiable technology of the form $Q = X_0^{1/4}X_1^{1/4}X_2^{1/2}$,

Table 1 Atemporal subtechniques to produce Q from (X_0;X_1,X_2) direct factors of labour, male and female labours

A : 1_0^A 33 of X_0^A 34 & 0_1^A 35 of X_1^A 36 & 0_2^A 37 of X_2^A 38 → $Q^A = 1^A$
* * * * *
B : 1_0^B 39 of X_0^B 40 & 1_1^B 41 of X_1^B 42 & 0_2^B 43 of X_2^B 44 → $Q^B = 3^B$
C : 1_0^C 45 of X_0^C 46 & 0_1^C 47 of X_1^C 48 & 1_2^C 49 of X_2^C 50 → $Q^C = 4^C$
D : 1_0^D 51 of X_0^D 52 & 1_1^D 53 of X_1^D 54 & 1_2^D 55 of X_2^D 56 → $Q^D = 5^D$

will after normalization of X_0 to unity, become more transparent two-dimensional $Q = X_1^{1/4}X_2^{1/2}$.)

To derive the post-Sraffian atemporal *local* functional relations—which will within each specified triangle prove to be (surprisingly!) *linear*—between Q and [X_0;X_1,X_2], if we wish to we can initially ignore any *break-even* equations such as those in Sraffa (1960). I will first demonstrate how a purely engineering approach can give my sought for Leontief-Sraffa *linear local* production functions, whose partial derivatives will, as non-spurious marginalisms, cogently pin down distributive real wage rates and Land's rent.

5. Pure-engineering full-employment Leontief-Sraffa locally linear non-spurious atemporal production functions

Figure 1 will usefully diagram the Table 1 scenario. When Crusoe (or society) has no positive Labours at all, we begin at the origin marked by A, because only A in Table 1 is then useable. A alone produces a paltry 1 of consumable corn. Knowing B and C would then be still not-yet-useable knowledge. At A all the corn harvest goes to landowners' rent.

At any point within the ΔABC, positive Labours now add to society's corn harvest. And in doing so, no longer does rent get all of the product. Society can optimally use all three (X_0;X_1,X_2) at their fully employed levels. When that gets done, rent no longer receives all of the Q^{ABC} product. Do not cry for the 'now-exploited' landowners. Why not? Socialists cried when the addition of capitals for Labour to work with generated a 'profit' or interest return to capitals—a 'vile subtraction' from Land's original deserving rent. Alas, all wrong. Landowners get more when workers sufficiently grow in numbers. However, throughout ΔABC land-rent of y_0^* will still remain at the low 1^A level of unit corn: this for the reason that some of the unit acreage still gets no Labours to work with and all acres must share their paltry rent rate. However, the newly created increment of total corn Q will be awarded competitively to males and females. Awarded equally? No. Table 1 shows that males are uniformly less productive than females; so y_1^* for males will be only three-quarters of y_2^* for females.

What ethical preceptor decided that possible violation of St. Thomas Aquinas' 'distributive justice'? The market has no heart and no conscience. Voracious would-be arbitragers by trial and error can clear all market supplies and demands solely at:

$$\left(y_0^*, y_1^*, y_2^*\right)^{ABC} = (1^*, 2^*, 3^*)^{ABC} \tag{8}$$

Why that? Because to the knowing eye, one perceives in Table 1 that everywhere inside ΔABC:

$$Q^{ABC} = 1^*X_0 + 2^*X_1 + 3^*X_2 \tag{9a}$$

$$\Delta Q/\Delta X_0 = 1^* \text{ an acre's rent} \tag{9b}$$

$$\Delta Q/\Delta X_1 = 2^* \text{ for male Labour's incremental productivity} \tag{9c}$$

$$\Delta Q/\Delta X_2 = 3^* \text{ for female Labour's incremental productivity} \tag{9d}$$

$$Q^{ABC} = \sum_0^2 y_j^* X_j^* = \sum_0^2 (\Delta Q/\Delta X_j) X_j (\text{QED}) \tag{9e}$$

Before looking for new $\Delta Q/\Delta X_i$ expressions, readers should test their own economist intuitions about comparative statics.

1. If both X_1 and X_2 rise while X_0 is constant, what *must* (!) happen to Land's y_0^{**}? Assuredly, if anything, rent must rise.

$$\left(y_0^{**}\right)^{BCD} \geq \left(y_0^*\right)^{ABC} \tag{10}$$

2. At the same time that y_0^{**} rent rises, it is a safe bet that *at least one* of the real wage rates soon falls. And maybe *both* y_1 and y_2 might fall, as in Table 1.

Actually, leisurely perusal of Table 1 nominates the marginalist's bet:

$$y_1^{**} = \Delta Q/\Delta X_1 = (\bar{5} - \bar{4})/(\underline{1} - \underline{0}) = 1^{**} < 2^* = y_1^* \tag{11a}$$

$$y_2^* = \Delta Q/\Delta X_2 = (\bar{5} - \bar{3})/(\underline{1} - \underline{0}) = 2^{**} < 3^* = y_2^* \tag{11b}$$

$$y_0^* = 1^* < y_0^{**} = 2^{**}, \text{ residually} \tag{11c}$$

A perfect take-home exam paper would deduce for each of ΔABC and ΔBCD the non-spurious locally linear production functions already recorded in Figure 1's legend:

$$Q^{ABC} = 1^{*}X_0 + 2^{*}X_1 + 3^{*}X_2,\ 0 < (X_1/X_0) + (X_2/X_0) < 1 \tag{12a}$$

$$Q_{BCD} = 2^{**}X_0 + 1^{**}X_1 + 2^{**}X_2, 1 < X_1/X_0 + X_2/X_0 < 2. \tag{12b}$$

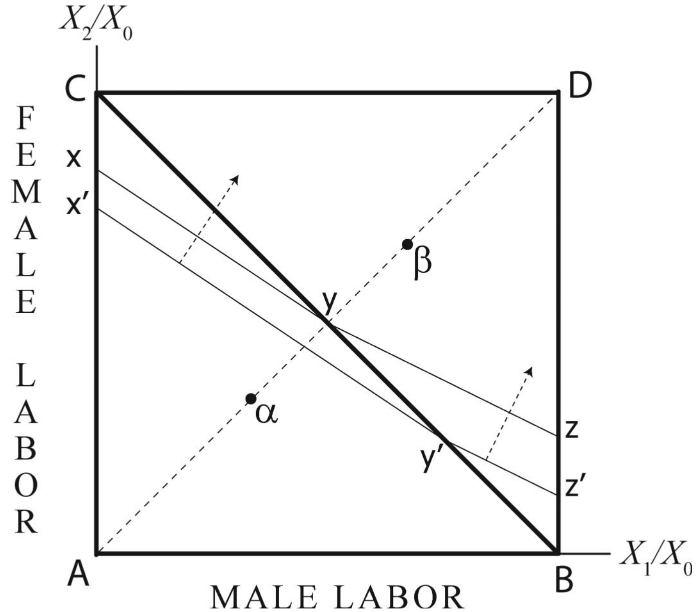

Figure 1 Where land with heterogeneous male and female laborers produce corn atemporally by alternative subtechniques. ***Notes:*** **Near the A origin, when Labour densities per acre of Land are light, full employment of inputs ($X_0 = 1; X_1, X_2$) can take place in, and only in, ΔABC. When populations of X_1 and X_2 crowd each acre further, full employment can be sustained only inside ΔBCD – where D outcompetes A in working with B&C sub-techniques. Inside each triangle, Table 1's data do generate linear (!)** ***non*****-neoclassical, non-spurious marginalisms: $Q^{ABC} = 1{*}X_0 + 2{*}X_1 + 3{*}X_2$; and $Q_{BCD} = 2{**}X_0 + 1{**}X_1 + 2{**}X_2$. In** ***qualitative agreement*** **with post-Clark neoclassical production functions, these pre-1870 Leontief-Sraffa limited-substitutability functions do comply with 1814 West-Malthus-Ricardo Laws of Diminishing Returns:** ***ceteris paribus*****, $\Delta^2 Q/\Delta Q_i^2 \le 0$, etc. Note that X′Y′ and XY are parallel straight lines when inside ΔABC** ***and*** **when inside ΔBCD. Note that the wider space between them in ΔBCD compared to ΔABC confirms Ricardo (1817) and Hicks (1939) diminishing returns: it does take** ***larger*** **factor-input increments to generate the** ***same*** **ΔQ when factor intensity (vectorally defined) is greater (QED).**

In the large because of technology's convexity, somewhat like revealed preference, there will have to be:

$$0 \geq (\Delta X_0)(\Delta y_0) + (\Delta X_1)(\Delta y_1) + (\Delta X_2)(\Delta y_2). \tag{13}$$

6. Sraffa-type break-even approach to atemporal equilibria

How might a Sraffian foot soldier try to determine the above correct $\left(y_0^*, y_1^*, y_2^*\right)$ and $\left(y_0^{**}, y_1^{**}, y_2^{**}\right)$ Ricardian distributional corn rent and corn wage rates? Armed only with Sraffa's (1960: part III) incomplete weapons, if clever he/she will try to find, for A and B and C—or for B and C and D—three (atemporal!) break-even equations such as the temporal break-even equations in Sraffa (1960: ch. 2).

Bravo! Here is what Table 1 does mandate. For ΔABC's interior points, $(X_1/X_0, X_2/X_0)^{ABC}$, with $P_{corn} = 1$ as numeraire:

$$A^* \ \bar{1} = y_0\underline{1}^A + y_1\underline{0}^A + y_2\underline{0}^A \tag{14a}$$

$$B^* \ \bar{3} = y_0\underline{1}^B + y_1\underline{1}^B + y_2\underline{0}^B \tag{14b}$$

$$C^* \ \bar{4} = y_0\underline{1}^C + y_1\underline{0}^C + y_2\underline{1}^C. \tag{14c}$$

For ΔBCD's interior points, $(X_1/X_0, X_2/X_0)^{BCD}$, one similarly writes:

$$B^{**} \ \bar{3} = y_0\underline{1}^B + y_1\underline{1}^B + y_2\underline{0}^B \tag{15a}$$

$$C^{**} \ \bar{4} = y_0\underline{1}^C + y_1\underline{0}^C + y_2\underline{1}^C \tag{15b}$$

$$D^{**} \ \bar{5} = y_0\underline{1}^D + y_1\underline{1}^D + y_2\underline{1}^D. \tag{15c}$$

Readers who have persisted with me this far can verify for themselves that only

$$(y_0, y_1, y_2)^* = (1^*, 2^*, 3^*)^{ABC} \tag{16a}$$

$$(y_0, y_1, y_2)^{**} = (2^{**}, 1^{**}, 2^{**})^{BCD} \tag{16b}$$

can clear all markets and kill off arbitragers' profit opportunities for Equations (14) and (15) (QED).

Much as the chicken pox virus can plague an adult's life forever with herpes, Sraffa's early antipathy toward general equilibrium slowed down his progress toward understanding non-spurious marginalisms. The last several paragraphs, with their $\Delta Q/\Delta X_i$ expressions, can perhaps constitute an expositional triumph to convert some borderline post-Sraffians. Skeptical Joan Robinson was a tougher mind, asking: 'Come, come, Samuelson, what can you hold constant when only one of numerous inputs gets varied?' Tables 1 and 2, with their pedagogically clever useful spray of zeros would only elicit her scornful veto. What she could not be made to understand—at least not by me—is that simultaneous equations do do the same job that those zeroes and ones could do.

Here is the BCD story in Table 1 and Figure 1, told by my merely solving the three full-employment linear equations for Land, male Labour and female Labour. This version eschews even mention of $\Delta Q/\Delta X_i$ expressions. (Readers can re-tell the ABC story once they do understand this BCD story.)

Any endowment vector, $(\hat{X}_0; \hat{X}_1, \hat{X}_2)^{BCD}$ *inside* ΔBCD can be fully employed when each of the following three linear relations is satisfied:

$$\text{Land}: X_0^B + X_0^C + X_0^D = \hat{1} = \hat{X}_0, \tag{17a}$$

$$\begin{aligned} \text{Male Labour}: X_0^B 1_1^B + X_2^C 0_1^C + X_0^D 1_1^D &= \hat{X}_1, \\ 0 < (X_1/X_0) + (X_2/X_0) &< 1 \end{aligned} \tag{17b}$$

$$\begin{aligned} \text{Female Labour}: X_0^B 1_2^B + X_0^C 1_1^C + X_0^D 1_2^D &= \hat{X}_2, \\ 1 < (X_1/X_0) + (X_2/X_0) < 2, 1 > (X_1/X_0) &< 1. \end{aligned} \tag{17c}$$

By subtracting (17b) from (17a), you deduce:

$$X_0^C = 1 - \hat{X}_1 \tag{17d}$$

By subtracting (17c) from (17a), you similarly deduce:

$$X_0^B = 1 - \hat{X}_2 \tag{17e}$$

Residually, then,

$$X_0^D = 1 - [1 - \hat{X}_1 + 1 - \hat{X}_2] = \hat{X}_1 + \hat{X}_2 - 1 \tag{17f}$$

Now the last three output entries on the right of Table 1 can show exactly what $Q = Q^B + Q^C + Q^D$ must be:

$$Q^{BCD} = Q^B + Q^C + Q^D \tag{18a}$$

$$= X_0^B 3^B + X_0^C 4^C + X_0^D 5^D \tag{18b}$$

$$= 3^B(1 - \hat{X}_2) + 4^C(1 - \hat{X}_1) + 5^D(\hat{X}_1 + \hat{X}_2 - 1) \tag{18c}$$

$$= 2^{**} + 1^{**}\hat{X}_1 + 2^{**}\hat{X}_2, \text{ for } \hat{X}^0 \equiv 1(\text{QED}) \tag{18d}$$

Note that selfish Darwinian competition wiped out any still 'arbitrage-able' profits, after the market was indeed led—as if by an Invisible Hand—to the 'maximal linear (non-spurious) first-degree-homogeneous production function' in Equation (18d). Whenever a Table like 1 (or like 2 to come) involves no visible *ceteris paribus* $\Delta Q/\Delta X_i$ experiments, that is of no consequence at all. Simultaneous equations à la (17) and (18) above generically generate the locally linear non-spurious Sraffian marginalisms.

Why bother to supplement the engineering approach by its equivalent Sraffa-type break-even approach? The main reason was to prepare readers for the temporal heterogeneous Ks cases to come. For them, as will be shown, Sraffa (1960)-type 'missing break-even equations' do generically fail to exist. Only in singular scenarios will his defined stationary states generate equality between Wheat's 'own rate of interest', r_1^*, and Iron's 'own rate of interest', r_2^*. Instead of a 'missing' equation, Sraffians will be faced with one break-even equation too many! Sad. But that is the way the cookie crumbles.[4]

7. Temporal heterogeneous capitals relate how to the atemporal $Q(t) = F[X_0(t); X_1(t), X_2(t)]$ model?

Mr. Etula has produced for me the following Leontief-Sraffa Table 2, whose likenesses and differences with atemporal Table 1 will become apparent to diligent readers.

Table 2 can provide for Sraffa (1960: part III), alternative sub-techniques that are known ways to produce gross Wheat output: call them **a**, **b**, and **c**. And it likewise postulates as known **A**, **B** and **C** alternative ways to produce gross Iron. For simplicity, Table 2 involves no joint products. Instead it has only 'circulating capitals', $K_1(t)$ and $K_2(t)$, that are

4 See Mathematical Appendix, which among other things does correct some remarks in Samuelson and Etula (2006a) alleging necessary equality of own rates of interest.

Table 2 Alternative ways for Labour & Wheat & Iron Inputs at t to produce at t + 1 Wheat & Iron gross outputs

Wheat : **a**	$\underline{1}^a$ of $L(t)^a$ & $\underline{0}_1^a$ of $K_1(t)^a$ & $\underline{1}_2^a$ of $K_2(t)^a \rightarrow Q_1(t+1)^a = \overline{4.2}^a$
b	$\underline{1}^b$ of $L(t)^b$ & $\underline{0}_1^b$ of $K_1(t)^b$ & $\underline{2}_2^b$ of $K_2(t)^b \rightarrow Q_1(t+1)^b = \overline{5.3}^b$
c	$\underline{1}^c$ of $L(t)^c$ & $\underline{0}_1^c$ of $K_1(t)^c$ & $\underline{3}_2^c$ of $K_2(t)^c \rightarrow Q_1(t+1)^c = \overline{6.35}^c$
Iron : **A**	$\underline{1}^A$ of $L(t)^A$ & $\underline{1}_1^A$ of $K_1(t)^A$ & $\underline{0}_2^A$ of $K_2(t)^A \rightarrow Q_2(t+1)^A = \overline{4.2}^A$
B	$\underline{1}^B$ of $L(t)^B$ & $\underline{2}_1^B$ of $K_1(t)^B$ & $\underline{0}_2^B$ of $K_2(t)^B \rightarrow Q_2(t+1)^B = \overline{5.3}^B$
C	$\underline{1}^C$ of $L(t)^C$ & $\underline{3}_1^C$ of $K_1(t)^C$ & $\underline{0}_2^C$ of $K_2(t)^C \rightarrow Q_2(t+1)^C = \overline{6.35}^C$

used up at t and must be replaced at t + 1. Any excess of $Q_i(t+1)$s above needed $K_i(t+1)$ to equal $K_i(t)$ will be positive final consumption of Wheat or Iron, namely, $C_i(t+1)$. By dimensional convention, I keep Labour's L always at unity:

$$L_1(t) + L_2(t) \underset{t}{\equiv} L(t) \equiv 1 \underset{t}{\equiv} L(t+1) \tag{19a}$$

As in Sraffa (1960: part I), readers can here *at first* assume that there is known only a *single* way of producing Wheat and a single way of producing Iron: say, **a&A**; or **a&B**; or.... In atemporal Table 1, when but *one* intertemporal technique had been known, distributive pricing was seen to be indeterminate. So it is here too in the temporal scenarios. Sraffa puts the matter nicely: we then face a 'missing equation'.

To coordinate with Sraffa's (1960: 11) price = costs exposition, I duplicate the **a&A** numerical data from Table 2 and write out Sraffa's two break-even equalities, which ensure that real prices, P_1/W and P_2/W, do exactly equal real unit costs calculated as the sum of input costs—$L(t)\&K_1(t)\&K_2(t)$ costs, where outlays on each of the Ks do always earn the same (safe!) rate of *interest* or *profit*, r:

$$\begin{aligned} 4.2^a P_1 &= 1^a W + 0_1^a P_1(1+r) + 1_2^a P_2(1+r) \\ 4.2^A P_2 &= 1^A W + 1_1^A P_1(1+r) + 0_2^A P_2(1+r) \end{aligned} \tag{19b}$$

Equations (19b) are manifestly but *two* equations in *three* unknowns: $(P_1/W, P_2/W; r)^*$. If a little birdie told us the true equilibrium value for any one of the three—say for r^*, or for one of $(P_j/W)^*$—then we Sraffians would face no 'missing equation' and could calculate (19b)'s possible distributive pricings.

8. Digression on 'a way not taken': Böhm-Fisher-Ramsey's intrinsic impatience time-preference to define missing equation(s)

For whatever reason, 1925 – 83 Sraffa revealed a general distaste for relying on subjective demand-tastes variables. None of his 1960 words relates to the classical and neoclassical objectively observable propensity of ordinary humans who may prefer a half loaf today to two loaves next year. By contrast, Irving Fisher or Pigou or Ramsey—or for that matter Nassau Senior or Böhm-Bawerk—usefully proposed scenarios where the typical family acted systematically as if it objectively applied, say, a five percent exponential per period discount parameter, $1/(1+\delta)^{\mathrm{T}} = \text{say } 1/1.05^{\mathrm{T}}$ discount factor to all economic metric values pertaining to T periods ahead in the future.

Then voila!, with the stroke of the pen, we have located the missing equation:

$$1 + \delta = (1 + \mathrm{r})^* = 1.05,\ \mathrm{r}^* = \delta = 0.05 \tag{19c}$$

We eclectic Sraffians, therefore, can put this $(1 + \mathrm{r})^*$ into Equations (19b) above. At sight Equations (19a) and (19b) enable one to write out for Ricardo *all* his needed competitive distribution parameters:

$$1 + \mathrm{r}^* = 1 + \delta = 1.05^*,\ \mathrm{r}^* = 5\% \text{ per period} \tag{19d}$$

Solving (19b) one deduces Ricardo's trade-offs:

$$\begin{aligned}(\mathrm{W/P_1})^* &= (3.2 - \mathrm{r})^* = (3.2 - \delta)\\ &= 3.20 - 0.05^* = 3.15^* = (\mathrm{W/P_2})^*\end{aligned} \tag{19e}$$

(The singular equality of real Iron wage rate and Wheat wage rate is of course solely due to the singular symmetries posited in Table 2.) Here I have followed Sraffa's convention of letting $\mathrm{W/P_j}$ stand for the real wage paid *post factum* to workers, at $\mathrm{t} + 1$ and not at t. Classical savants thought it more realistic to have rentiers 'advance' to workers their wage at time t. And, of course, on such advances rentiers would insist on the same r* interest rate as is earnable on *all* of their non-wage investment outlays.

Instead of plucking exogenous δ out of the air, a sage Modigliani could utilize his excellent life-cycle saving scenario, where supply and demand between (1) retired folk of all ages and (2) working-age folk of all ages, would just balance out at a market-clearing r_1^*. In such a special model a society could even be a strictly egalitarian classless society. (Also, there could be multiple equilibria.)

See Mathematical Appendix for a generalization of Ramsey's (1928) *scalar* capital flow model of optimal saving to Leontief-Sraffa discrete-time paradigms of *heterogeneous* capitals. For positive or zero δ, it is differences between r_1 and r_2 that get wiped out in the asymptotic final dynamic equations where $r_1 = \delta = r_2$.

9. Piero's preferred way

Sraffa (1960: part III) went some limited steps toward seeking missing equations by another route—namely, by combining a triad such as **a&b&A** or **b&A&B** sub-technologies. Briefly, too briefly, I will presently sketch here how use of *four* sub-techniques simultaneously—say **a&b&A&B** or **b&c&A&B**—could generate *non*-neoclassical marginalisms that are definitely non-spurious and that do maximize permanent levels of final Wheat or final Iron. To do this in a few limited words will force me to temporarily only sketch some genuine stationary-state subtleties.[5]

Table 2's data do not tell their own story. Those technological data, *when augmented by exogenous demand-tastes data* of several different contemplated Robinson Crusoes, can be shown to lead to quite different alternative post-Sraffian distributions-of-income equilibria.

Consider a Crusoe who wants only Wheat as a final utility good. That is but the first of many different possible patterns of taste. He of course differs from a second Crusoe who wants only Iron as a final good.

A third demand pattern worth exploring could be for a Crusoe who, à la J.S. Mill (1848), always spends any of his income fifty – fifty percent on the two goods. Or spends two-thirds on Wheat and one-third on final Iron; or spends one-third on Wheat.... A fourth demand pattern could be for a Crusoe who has symmetric *linear* utilities. He would allocate his unit $L = 1$ optimally among his (K_1,K_2) input endowments so as to maximize $C_1 + C_2$ consumptions.

A fifth demand pattern is for a Crusoe who seeks as final consumption a fixed dose of both Wheat and Iron. His cornered utility function could be, say, $\mathrm{Min}[C_1,C_2]$. For him 3 of Wheat and 3 of Iron would be indifferent to 3 of Wheat & 300 of Iron; and be indifferent to 300 of Wheat & 3 of Iron. Almost certainly, given any flexibility of input allocation, this Crusoe will equate consumptions for Wheat and Iron: $C_1 = C_2$.

5 Sraffa (1932), in his polemic against Hayek (1931), importantly originated consideration of 'own rates of interest in Wheat,' r_1^*, and 'own rates in Iron,' r_2^*. Keynes (1936: ch. 17) comments on this somewhat obscurely, as pointed out in Pigou (1936) and Samuelson (1937, 1939). Generically, for most exogenous $(K_1/L,K_2/L)$ endowments, $r_1^* \neq r_2^*$! So to speak this serves as a signal for the system to *leave* the stationary state and proceed with generalized Ramsey (1928) dynamics. See the present Mathematical Appendix that handles for Ramsey heterogeneous capitals produced over finite discrete time periods, t and t + 1.

Figure 2, which is perhaps the most important part of this article, does present diamond quadrilaterals near the main diagonal that apply Table 2's sub-technologies to a Crusoe with the above fifth pattern of equal dose Iron – Wheat tastes. I accepted Erkko Etula's nomination of this pattern, because it is in a genuine sense the demand pattern *most different* from the post-1870 *neoclassical* differentiable utilities of Jevons-Walras-Menger. Also, it does best utilize the simplifying skew symmetries of Table 2 and Figure 2. However, the $C_2 \equiv 0$ case is perhaps the easier one to talk about initially.

10. The purely engineering equations of competitive equilibrium

The competitive auction market has no mind; no heart; no will. What drives it is the selfish desire of input owners to end up with most possible command over Wheat and Iron outputs. In stationary equilibrium solely when four sub-techniques come into use simultaneously will nothing be left on the table for eager myopic arbitragers to scoop up?

What I am sketching is what a second edition of Sraffa (1960) might have included in a new part III or IV. Generically, two heterogeneous capitals achieve maximal permanent outputs of goodies only when *four* viable sub-techniques get used. With techniques feasibly adjusted to the exogenous endowment vector, supply and demand market clearing will mandate that unit supply of L gets divided into (L^a, L^b, L^A, L^B) uniquely so as to leave none of the three inputs $\left(L = 1, K_1^e; K_2^e\right)$ *unemployed* while at the same time consumers' spending evokes the gross Qs that permit maximal desired Cs.

I now spell out here the determining linear equations, necessary and sufficient, for characterizing competitive distribution equilibrium for 1750 – 1870 classical regimes and 1960 – 2007 non-neoclassical regimes. Figure 2's α point inside a′bAB has exact $(K_1/L, K_2/L)$ coordinates of $(0.3, 1)^{\alpha}$. For β, coordinates are $(0.5,\ 1.5)^{\beta}$. Side by side, here are the respective four linear relations:

For α :	For β :	
$L^a + L^b + L^A + L^B = 1$	$L^b + B^c + L^A + L^B = 1,$ L fully employed	(20a)
$\underline{0}_1^a L^a + \underline{0}_1^b L^b + \underline{1}_1^A L^A + \underline{2}_1^B L^B = 0.3_1^e$	$\underline{0}_1^b L^b + \underline{0}_1^c L^c + \underline{1}_1^A L^A + \underline{2}_1^B L^B = 0.5_1^e,$ K_1 fully employed	(20b)

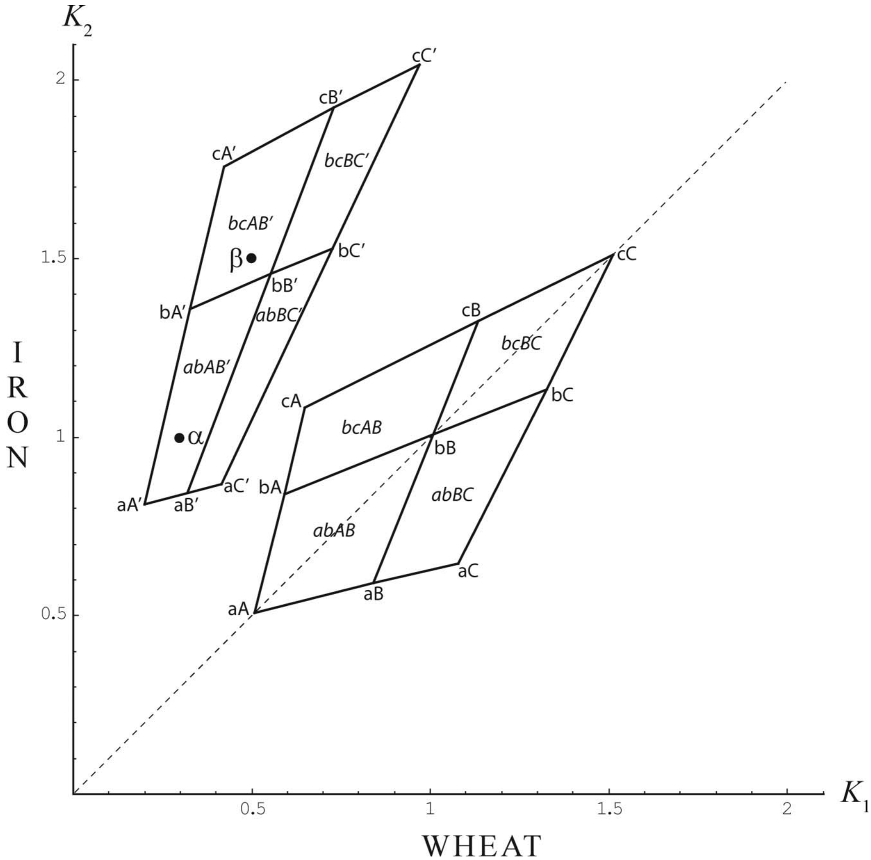

Figure 2 Where heterogeneous wheat and iron are produced by themselves and labour. ***Notes*****: The four diamond-shaped quadrilaterals northwest of the main diagonal do map *all* the full-employment endowments that can sustain stationary equilibrium under Table 2's known technological data when Wheat is Crusoe's *sole* desired final consumption good: $C_1(t+1) > 0 \equiv C_2(t+1)$. When Crusoe's demand tastes have changed so that C_1 and C_2 are to be equal and be maximal, Table 2's data will generate the four diamonds near the main diagonal: only inside those four are the fully and permanently employed endowment points.**

To understand Leontief-Sraffa non-spurious marginalisms, it will suffice to contemplate just two of the eight diamonds: say, a point like α in $a'bAB$; and a point like β in adjacent $b'cAB$. At each such point, stationary maintained equilibrium is reached by mindless avaricious Darwinian competitors. Equilibrium is reached only where the four allocated Labour fractions achieve permanent full-employment of the three total inputs; and satisfy also Crusoe's objectively specified $C_1 > 0 \equiv C_2$ demand conditions. Equations (20), simple linear equations, do suffice to determine unique $(L^a,L^b,L^A,L^B)^*$ fractions for α; and for β, unique $(L^b,L^c,L^A,L^B)^{**}$ fractions. Entering such a known foursome into the indicated rows of input and

$$\begin{array}{ll} \underline{1}^{a}_{2}L^{a}\ \underline{2}^{b}_{2}L^{b} + \underline{0}^{A}_{2}L^{A} & \underline{2}^{b}_{2}L^{b} + \underline{3}^{c}_{2}L^{c} \\ \quad + \underline{0}^{B}_{2}L^{B} = 1^{e}_{2} & \quad + \underline{0}^{A}_{2}L^{A} + \underline{0}^{B}_{2}L^{B} = 1.5^{e}_{2}, \\ & \quad K_2 \text{ fully employed} \end{array} \tag{20c}$$

$$\begin{array}{ll} \overline{4.2}^{A}L^{A} + \overline{5.3}^{B}L^{B} = 1^{e}_{2} & \overline{4.2}^{A}L^{A} + \overline{5.3}^{B}L^{B} = 1.5^{e}_{2}, \\ & \quad C_2(t) \equiv 0 \end{array} \tag{20d}$$

Solved out by any of many elementary substitutions, and after the fractional Ls are entered into Table 2's appropriate rows, one finds spelled out the two locally linear Leontief-Sraffa production functions reported in Figure 2's lengthy legend. At last, Joan Robinson's query: 'When you claim to measure $\Delta C_1/\Delta L$ or $\Delta C_2/\Delta K_i$, what variables are you controlling in your alleged *ceteris paribus*; and which variables are varying?' I write out the appropriate answer for her:

For abAB endowments:

$$C_1 + 1^{*}C_2 = 3.1^{*}L + 0.1^{*}K_1 + 1^{*}(0.1)^{*}K_2 \tag{20e}$$

$$\partial C_1/\partial L \equiv \Delta C_1/\Delta L = 3.1^{*} = (W/P_1)^{*},\ \text{real Wheat wage} \tag{20f}$$

$$\partial C_1/\partial K_1 \equiv \Delta C_1/\Delta K_1 = 0.10^{*} = \text{own Wheat interest rate } r_1^{*} \tag{20g}$$

$$-\ \partial C_1/\partial C_2 = -\Delta C_1/\Delta C_2 = 1^{*} = (P_2/P_1)^{*} \tag{20h}$$

$$\partial C_2/\partial K_2 = \Delta C_2/\Delta K_2 = r_2^{*} = (\Delta C_1/\Delta K_2)\pi^{*} \tag{20i}$$

output numbers in Table 2, we do end up with non-spurious marginalisms: $(C_1 + \pi^{*}C_2)^{abAB} \equiv (C_1 + \pi^{*}0) = (C_1 + [P_2/P_2]^{*}0) = (C_1 + 1^{*}0) = \rho_0^{*}L + \rho_1^{*}K_1 + \rho_2^{*}K_2 = 3.1^{*}L + 0.10^{*}K_1 + 0.10^{*}K_2 \equiv (W/P_1)^{*}L + r_1^{*}K_1 + \pi^{*}r_2^{*}$. Here $\pi^{*} \equiv 1$ is due solely to singular (!) skew symmetry. *Generically* at β (the more interesting case), $(L^{b},L^{c},L^{A},L^{B})^{**}$ translated into Table 2 ends us up with $(C_1 + 1.03^{**}0)^{bcAB} = 3.2^{**}L + 0.1355^{**}K_1 + 1.03^{**}(0.0172)^{**}K_2 \equiv (W/P_1)^{**}L + r_1^{**}K_1 + (P_1/P_1)^{**}r_2^{**}$.

As expected, more of both K_1/L and K_2/L does raise real wage from 3.1* to 3.2. Instead of lowering both r_1^{**} and r_2^{**}, going to β from α does lower Iron's own interest rate from ten percent per period to 1.72 percent per period. However, as economists' intuition permits, Wheat's own interest rate actually rises from ten percent to 13.55 percent at β. To test and confirm comprehension, readers can put their own new α' and β' points in any other adjacent diamonds, thereby deducing similar comparative statics.**

The moral of Table 2's tale is that generically r_1^{*} and r_2^{*} will differ (QED).

The point that needs stressing is how very much demand tastes do matter. A secondary point is important too: a pro-Sraffian, who believes strongly that the world empirically has but few viable alternative sub-technologies, ought to expect distributive shares to jump around volatilely in a way that econometricians do not find to occur. Bowley's Law of fairly constant distributive shares is a reported econometric approximation, but even the systemic trends away from it do seem remarkably stable.

11. Epilogue

I had hoped on this occasion to provide a fairly complete synthesis of the Master Function methodology of recent Samuelson and Etula (2006a,b,c) and Etula papers in the pipeline. However, resistance from various journals' peer reviewers slowed down the publication programme, as did compulsory service in the Finnish army for MIT-Harvard graduate student Erkko Etula.

My final words are directed toward the unusual phenomenon of a lone autodidactic researcher who for a third of a century occupied his limited spare time toward one grand original purpose. In the annals of the many corners of science and scholarship, there are at least a few such known cases. Far fewer though are the subset who did in the end succeed in adding significantly to posterity's canon of agreed-upon wisdoms.

Piero Sraffa has been a notable case in point. From age 27 years onward, he became preoccupied with the complexities of intertemporal capital theory. This began before he was singled out by the Royal Society to compile the definitive editions of David Ricardo's papers. The Great Depression and Second World War (during which he was interned in Britain as an alien from an enemy country) interrupted and slowed down his major theoretical research programme. But still he persisted.

The great Albert Einstein offers some limited parallelisms. From 1905 to 1925 again and again Einstein initiated revolutions on many different physics fronts: special relativity, Brownian motion, post-Planck quantum physics, general relativity (this latter crowned him as successor to Isaac Newton himself!).

And then, almost as an anti-climax, in the last third of his life, his past-earned self-confidence led him away from the mainstream of 1930–2007 physics. Tirelessly, and with able young collaborators, Einstein pursued his own paths to try to unify relativity and quantum theory. The little progress he made turned out to be definitely not in the direction that actual living physics was going. His was a gigantic struggle, but also in it there was a definite element of pathos. His attempted refutations (addressed to Niels Bohr) of an ultimate probability basis to physical laws involved ingenious thought experiments—virtually *reductio ad absurdum* experiments. Alas, precisely what he expected readers to regard as ridiculously implausible are today's well-documented 'entanglement' phenomena that may generate the future's miniature powerful computers.[6]

One thinks too of Ramanujan, the poor and tubercular Madras clerk who was discovered through the mail by Trinity College's great mathematician, G.H. Hardy. Together they made beautiful transcendental music that Hardy could not possibly have done alone. In a burst of romanticism, Hardy once hypothesized that Ramanujan's environmentally induced lack of much of established maths freed his mind to soar where the academy never dreamed of. However, later, in a more sober moment, Hardy recanted, admitting in effect: 'How much more glorious Ramanujan's accomplishment could have been had he had the good health and full advantages of a superior training in all of modern mathematics'.

By temperament, Piero Sraffa preferred to originate in his own way. Help he did get from Frank Ramsey. (One 1928 note someone sent me from Ramsey to Sraffa sketched out matrix equations of not-yet-discovered Dantzig (1963) linear programming and Kuhn-Tucker non-linear concave programming.) Cambridge's great mathematician Besicovich also was an acknowledged helper. But it was learned from Mrs. Besicovich how frustrating it was to give help to a friend who never fully revealed what his targets were. In my small way (and long before I learned only at the 1958 IEA Corfu meeting that Sraffa was about to publish a book on capital theory), I would often say in talking with him things like: 'All you need for this are the Hawkins and Simon (1949) determinant inequalities to assure a surplus economy'. He brushed aside any such prattle about this and other well-known Kuhn-Tucker concave programming dualities or Richard Bellman intertemporal generalizations of the calculus of variations. I am sure he never cracked the pages of the Dorfman *et al.* (1958) book that I sent him. Understandably, he wanted to do his way whatever he was to do.

I respect and salute Piero Sraffa. He added colour, but beyond colour he did add light to the not-so-dismal science of economics.

Like Pliny the Younger of Rome, I have to apologize for the lengthiness of this analysis on the grounds that I lacked the time to make it shorter.

6 Richard Feynman, Einstein's worthy successor, has termed 'entanglement' as the *essential* weirdness of quantum theory; weird and inexplicable but irrefutably present.

Acknowledgements

Without Harvard-MIT student Erkko Etula, I could not have written this paper. Were it not for the need to make so many Sraffa-Samuelson *ad hominem* remarks, this could have been a joint Samuelson and Etula article. His laptop computer ground out the many diagrams and numerical wages and rents and input allocations. Editor Janice M. Murray triumphed over a many-time revised, untidy manuscript. As usual, Robert Solow pruned errors from my scribbles. All imperfections have been mine alone and inevitably their number will not be zero. All the definite faults herein trace to my own logical and empirical imperfections.

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Mathematical Appendix

Post-Ramsey (1928) optimal saving for heterogeneous capitals: Neoclassical technologies and Leontief-Sraffa limited-substitutability technologies

Ramsey (1928) used the following kind of model for Robinson Crusoe's optimal time profile of 'abstaining-saving':

$$\mathrm{Max}_{K(t)} \int_0^\infty e^{-\delta t} U\{F[K(t)] - \alpha K(t) - K'(t)\},\ \delta \geq 0,\ \alpha > 0 \qquad \text{(A1)}$$

where α is a positive durability-of-K parameter and where positive δ measures Crusoe's objectively observable 'impatience' or 'systematic time preference' parameter.

Were Crusoe's δ to be zero, starting from low initial K(0), he would opt to 'abstain' from some current C(t) in order to attain growth in K(t) toward the optimal $K(\infty)^g$ 'golden-rule K^g,' which will give him permanent (perpetual) maximal $C(\infty)^g$. In agreement with Schumpeter's (1912) heuristic conjectures, at $K(\infty) = K^g$, the safe interest rate $r(\infty)^g = 0$ will denote euthanasia of the rentier capitalists—until some new Schumpeterian innovations come along.

For Crusoes with positive δ impatience, less is saved in initial and later years, so that the ultimate $[K^\delta, C^\delta]$ asymptote is accordingly lower. K^δ and r^δ each relate *inversely* while r^δ grows with δ. I skip Ramsey's cogent proofs.

Ramsey (1928) used a neoclassically differentiable concave F[ ] production function with *scalar* K. Samuelson and Etula (2006b) also dealt with scalar K but in the Leontief-Sraffa context of *limited substitutability.* I omit the cogent proofs and explications to leave room here for the scenario with *heterogeneous* Wheat and Iron capitals.

Both for the differentiable neoclassical technology and the Leontief-Sraffa limited substitutability technology, I begin with the following definable Master Function. It is for a Crusoe whose objective demand tastes are for Wheat only as a final good—for $C_1(t+1)$. For Crusoe, always $C_2(t+1) \equiv 0$. Therefore:

$$C_1(t+1) = -K_1(t+1) + F[L(t); K_1(t), K_2(t); K_2(t+1) + 0] \qquad \text{(A2a)}$$

$$\equiv M[K_1(t), K_2(t); K_1(t+1), K_2(t+1)], \quad \text{for } L(t) \equiv 1 \qquad \text{(A2b)}$$

$$\partial M/\partial K_i(t) > 0 > \partial M/\partial K_i(t+1), i = 1, 2. \qquad \text{(A2c)}$$

Equations (A2) hold both for neoclassical functions like Cobb-Douglas, or for any Leontief-Sraffa technology like that in Table 2, where their M function will lack two-sided partial derivatives on definable boundaries of regions in the $[K_1/L, K_2/L]$ two-dimensional plane.

This appendix's ultimate purpose is to deduce that almost all $[K_1/L, K_2/L]$ stationary states will generate *unequal* 'own-Wheat and own-Iron rates of interest.' That is:

$$\partial K_1(t+1)/\partial K_1(t) = 1 + r_1 \neq 1 + r_2 = \partial K_2(t+1)/\partial K_2(t). \qquad \text{(A3)}$$

Our Crusoe seeks to maximize over an infinite lifetime, from $t = 0$ to $t = \infty$, the present value of all his future *discounted* concave utilities:

$$\text{Max} \sum_{t=0}^{\infty} [1+\delta]^{-t} U\{C_1(t+1)\}, U'\{\ \} > 0 > U''\{\ \} \tag{A4a}$$

$$\text{Max} \sum_{t=0}^{\infty} [1+\delta]^{-t}\ U\{M[K_1(t-1), K_2(t-1); K_1(t), K_2(t)]\}; L(t) \equiv 1. \tag{A4b}$$

For this infinite sum to be maximal, Crusoe must for every T pick $[\mathbf{K_1(T)},\mathbf{K_2(T)}]$ to optimize the sum of the following two adjacent expressions:

$$\underset{K_1(T),K_2(T)}{\text{Max}} \ldots + [1+\delta]^{-T} U\{M[K_1(T-1), K_2(T-1); K_1(T), K_2(T)]\} \tag{A4c}$$

$$+ [1+\delta]^{-T-1} U\{M[K_1(T), K_2(T); K_1(T+1), K_2(T+1)]\} + \ldots \tag{A4d}$$

$$= \text{for short}, \underset{K_1,K_2}{\text{Max}} [1+\delta]^{-T} \Phi(\underline{K}_1, \underline{K}_2; K_1, K_2;\ \overline{K}_1, \overline{K}_2]. \tag{A4e}$$

A necessary condition for such a maximizing ‘extremal’ path is that:

$$\{\partial/\partial K_i(\mathbf{T})\}[1+\delta]^{-T} \Phi(\underline{K}_1, \underline{K}_2; K_1, K_2;\ \overline{K}_1, \overline{K}_2) = 0, i = 1, 2. \tag{A4f}$$

For $i = 1,2$, Equation (A4f) boils down after a cancellation of the $[1+\delta]^{-T}$ factor to:

$$1+\delta = -\frac{U'\{M[K_1(T), K_2(T); K_1(T+1), K_2(T+1)]\} M_i[K_1(T), K_2(T); K_1(T+1), K_2(T+1)]}{U'\{M[K_1(T-1), K_2(T-1); K_1(T), K_2(T)]\} M_{2+i}[K_1(T-1), K_2(T-1); K_1(T), K_2(T)]} \tag{A5a}$$

Recall that Equation (A5a) uses the subscript notation:

$$\partial f(x_0, x_1, \ldots, x_N)/\partial x_i = f_i(x_0, x_1, \ldots, x_N), i = 0, 1, \ldots, N \tag{A5b}$$

Equation (A5a) gives twentieth century Kuhn-Tucker-Bellman extremal conditions that are analogous to eighteenth century Euler-Lagrange extremal conditions for standard calculus of variations problems.

In Crusoe's asymptotic terminal stationary state, the U′ { } terms in Equation (A5a) will cancel out, and the equations become:

$$1+\delta = -\frac{M_i\left[K_1^g, K_2^g; K_1^g, K_2^g\right]}{M_{2+i}\left[K_1^g, K_2^g; K_1^g, K_2^g\right]}, i = 1, 2 \quad \text{(A6a)}$$

$$= \partial K_i(t+1)/\partial K_i(t) = 1 + r_i = 1 + \delta \text{ (QED)} \quad \text{(A6b)}$$

The dynamic generalized vectoral 'deepening of capital' story deduced here for both neoclassical and Leontief-Sraffa heterogeneous (!) capitals thus does affirm precisely what Joan Robinson used to deny as mere trickle-down flap-doodle capitalistic apologetics. She expected only the likes of Nassau Senior, Eugen von Böhm-Bawerk, A.C. Pigou and Irving Fisher to believe such nonsense. I invite Sraffian friends to audit unmercifully these Ramsey-Samuelson syllogisms as applied to Robinson (1956) and Sraffa (1960).

This appendix will be complete once I deduce generically that only a razor's edge of $[K_1/L, K_2/L]$ endowments can generate *equal* 'own Wheat' and 'own Iron' interest rates.

First the easy, old-hat, neoclassical case. Consider the following generic specimen of a neoclassical Master Function:

$$C_1(t+1) + K_1(t+1) + 0 + K_2(t+1) = L(t)^{\frac{1}{2}}K_1(t)^{\frac{1}{2}} + L(t)^{\frac{3}{4}}K_2(t)^{\frac{1}{4}} \quad \text{(A7a)}$$

This specimen happens to involve the kind of Sraffa (1960: part II, III) joint production of Wheat and Iron by the same L(t).

For $L(t) \equiv 1$, Equation (A7a) becomes:

$$C_1(t+1) + K_1(t+1) + K_2(t+1) = K_1(t)^{\frac{1}{2}} + K_2(t)^{\frac{1}{4}} \quad \text{(A7b)}$$

$$\therefore \frac{\partial K_1(t+1)}{\partial K_1(t)} = \frac{1}{2}K_1(t)^{-\frac{1}{2}}, \frac{\partial K_2(t+1)}{\partial K_2(t)} = \frac{1}{4}K_2(t)^{-\frac{3}{4}} \quad \text{(A7c)}$$

$$\frac{1}{2}K_1(t)^{-\frac{1}{2}} \neq \frac{1}{4}K_2(t)^{-\frac{3}{4}}, \text{generically.} \quad \text{(A7d)}$$

However, for what singular endowment point $\left(K_1^e/L, K_2^e/L\right)$ will equality of own rates, $r_1 = r_2$, be possible? My post-Ramsey dynamics mandates solving for:

$$\frac{1}{2}K_1^{-\frac{1}{2}} = \frac{1}{4}K_2^{-\frac{3}{4}} \quad \text{(A7e)}$$

Therefore, equal r's can occur only on the razor's edge:

$$\left(K_1^e/L\right) = 4\left(K_2^e/L\right)^{\frac{3}{2}} \tag{A7f}$$

For a similar proof of inequality of own rates when technologies are of limited substitutability, any motivated Sraffian can specify at random **a**&**b**&**C**&**D** alternative numerical known ways of producing permanently positive C_1 and C_2.

Generically, this will define locally the following (linear!) Sraffian production function:

$$\begin{aligned}&[C_1(t+1)+K_1(t+1)]+\pi^*[C_2(t+1)\\&\quad+K_2(t+1)]=b_0^*L(t)+b_1^*K_1(t)+b_2^*K_2(t)\end{aligned} \tag{A8a}$$

If, and only if, the following singular equality holds, will:

$$b_1^* = b_2^*/\pi^* \tag{A8b}$$

$$= (1+r_1)^* = (1+r_2)^* \tag{A8c}$$

In the generic case, almost never will this happen. Table 2, contrived for me artfully by Erkko Etula's L.P. Dantzig programme, exhibits in Figure 2 four diamond-shaped regions: two diamonds, e.g. the ones that surround the 45° diagonal in the $(K_1/L, K_2/L)$ space, do exhibit equality $r_1^{abAB} = r_2^{abAB}$ because of imposed skew-symmetry. However, inside the other two diamonds, symmetry is broken and (generically):

$$b_1^* \neq b_2^*/\pi^* \tag{A8d}$$

Note that changing every Table 2 coefficient at random by ever so little as $+\text{-}\frac{1}{100}$ will generically negate (A8c)'s singular equality (QED).

What can suffice to defang differences in own rates

It could be the case that a rational Robinson Crusoe systematically applies a δ_1 impatience parameter for Wheat consumptions *different* from his δ_2 impatience parameter for Iron consumptions.

Specifically, replace Crusoe's (A4) by the following:

$$\mathrm{Max}\sum_{0}^{t}\left[1+\frac{1}{10}\right]^{-t}U_1\{C_1(t+1)\}+\sum_{0}^{t}\left[1+\frac{2}{10}\right]^{-t}U_2\{C_2(t+1)\} \tag{9a}$$

The simplest example to explicate the point could be the following Leontief-Sraffa or neoclassical Master Function that holds when each Q_j and C_j uses only itself as an input along with a fixed-supply Labour specific only to it: say female Labour, $L_1 = 1$ for Wheat and male Labour, $L_2 = 1$, for Iron.

This implies the following Master Function:

$$Q_1(t+1)+Q_2(t+1)=F^1[K_1(t)]+F^2[K_2(t)] \tag{9b}$$

Were this to obtain, then in Crusoe's ultimate steady state, he will end up with:

$$r_1(\infty)=\frac{1}{10}<r_2(\infty)=\frac{2}{10} \tag{9c}$$

Applying this loophole to the 1776–2006 time preference literature, I can contrive legitimacy for *any* $(K_1/L, K_2/L)$ in a dense region as a stationary state.

From the standpoint of behavioural economics introspection, why could not my time preference for Wheat and for Iron consumptions significantly differ? Most people's time preference for, say, dancing probably does exceed their time preference for jogging or doing the dishes (QED).

Abstract

Proofs are given that only singularly can real 1750–2007 competitive price ratios be 'natural', in the sense of being invariant under changes in demand tastes. Proofs are given that both 1750–1870 discrete technologies or 1890–2007 continuum technologies, *with convexity properties sufficient for arbitrage-proof supply-demand equilibria,* will be 'intertemporally Pareto optimal', immune to leaving any deadweight (inefficient) losses on the table. Sraffa (1960), ignoring the vast post-1945 linear and non-linear programming mathematical literature of Danzig, Kuhn-Tucker-Bellman, von Neumann, Ramsey literature does not quite arrive at attainable

distribution solutions. Where it tolerates *increasing* or *decreasing* returns to scale, there can be no *competitive* equilibria. When its matrix equations do obey first-degree-homogeneous functions, the book's stress on Basics or non-Basics is an irrelevancy leading to bizarre novel interpretations of Ricardo.

Old age overtakes us all. Alas, Sraffs's proposed critique of twentieth century political economy we will never be able to know.

Keywords

Non-spurious marginalisms for limited-substitutability or smooth differentiable technologies, 'Master Functions' (cornered or smooth), scales-return constancy for competition, generic *inequality* of *own rate of interest*!